

HOW SCALING SYMMETRY SOLVES A SECOND-ORDER DIFFERENTIAL EQUATION

Sidney Bludman* and Andres Guzman†

Departamento de Astronomía, Universidad de Chile, Santiago, Chile

Dallas C. Kennedy‡

(Dated: December 4, 2012)

While not generally a conservation law, any symmetry of the equations of motion implies a useful reduction of any second-order equation to a first-order equation between invariants, whose solutions (first integrals) can then be integrated by quadrature (Lie's Theorem on the solvability of differential equations). We illustrate this theorem by applying scale invariance to the equations for the hydrostatic equilibrium of stars in local thermodynamic equilibrium: Scaling symmetry reduces the Lane-Emden equation to a first-order equation between scale invariants u_n, v_n , whose phase diagram encapsulate all the properties of index- n polytropes. From this reduced equation, we obtain the regular (Emden) solutions and demonstrate graphically how they transform under scale transformations.

PACS numbers: 45.20.Jj, 45.50.-j, 47.10.A-, 47.10.ab, 47.10.Df, 95.30.Lz, 97.10.Cv

Contents

I. SYMMETRY REDUCES THE ORDER OF ANY SECOND ORDER ODE	1
II. HYDROSTATIC EQUILIBRIUM OF SELF-GRAVITATING SPHERES	2
III. SCALE TRANSFORMATIONS ON SOLUTIONS OF THE LANE-EMDEN EQUATION	3
IV. PHASE DIAGRAM FOR ALL FIRST INTEGRALS	5
V. CONCLUDING SUMMARY	6
References	6

I. SYMMETRY REDUCES THE ORDER OF ANY SECOND ORDER ODE

Lie showed how the invariance of a second-order ordinary differential equation (ODE) under a point symmetry leads constructively to a first-order ODE plus a quadrature. If the symmetry were a variational symmetry of the Action Principle, Noether's well-known theorem would lead to a conservation law [1–3].

We consider the *scaling symmetry* $\xi \rightarrow A\xi$, $\theta_n(\xi) \rightarrow \theta_{nA}(\xi)$ of the second-order ODE *Lane-Emden equation* (Section III) which describes the hydrostatic equilibrium of a gaseous sphere or star in local thermodynamic equilibrium (Section II). Because this is not a variational symmetry, but only a symmetry of the Lane-Emden equation, scaling symmetry leads only to a *non-conservation law* [4, 5]. This is a first-order differential equation for scale invariants u_n, v_n , which can be solved for $v_n(u_n)$ for given boundary conditions (Section III). From these first integrals, quadrature finally leads to solutions of the original second-order equation (Section IV).

Sections II and III will consider only *regular solutions* which have density finite at the origin and apply only to complete polytropes. Section IV will generalize to *irregular solutions* which have densities infinite at the origin (F-solutions) or vanish away from the origin (M-solutions) and can apply only to stellar envelopes.

*Electronic address: sbludman@yahoo.com; URL: <http://www.das.uchile.cl/~sbludman>

†Electronic address: aguzman@das.uchile.cl

‡Electronic address: dalet@stanfordalumni.org; URL: <http://home.earthlink.net/~dckennedy>

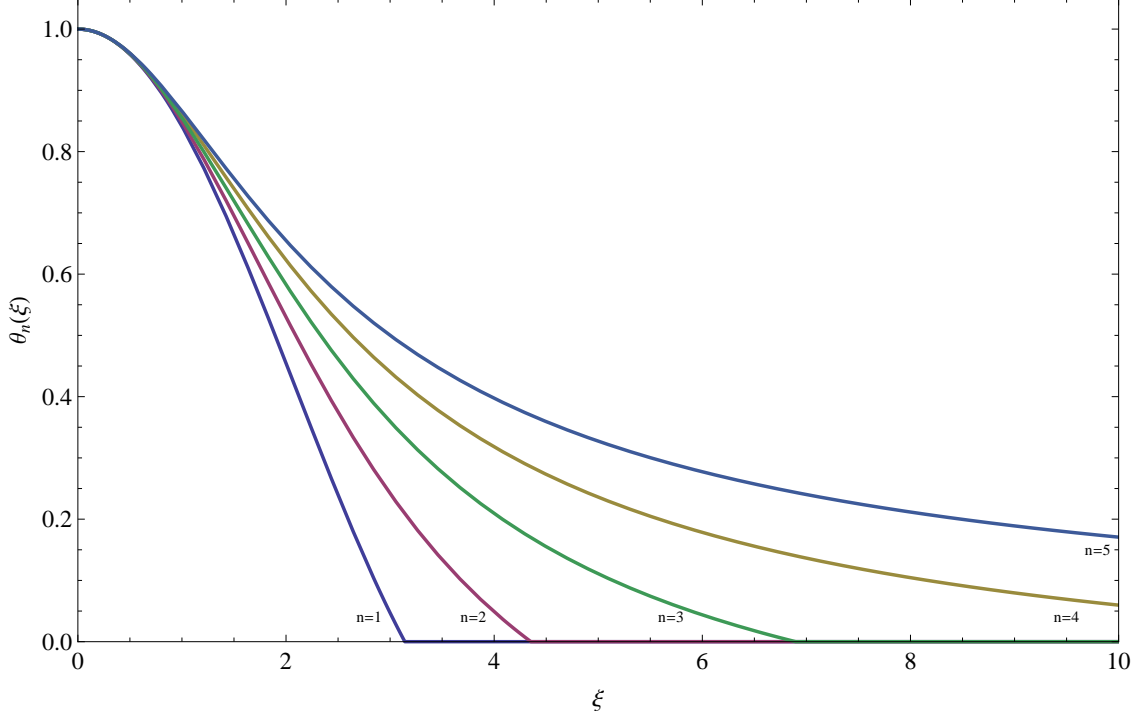


FIG. 1: Emden Functions (Regular Solutions of the Lane-Emden Equation) of order $n=1,2,3,4,5$

II. HYDROSTATIC EQUILIBRIUM OF SELF-GRAVITATING SPHERES

An adiabatic sphere in hydrostatic equilibrium obeys the equations of equilibrium between gravitational and internal (pressure) forces and of mass continuity

$$-dP/\rho dr = Gm/r^2, \quad dm/dr = 4\pi r^2 \rho, \quad (1)$$

where the local pressure, mass density, and included mass $P(r)$, $\rho(r)$, $m(r)$ depend on radius r . In terms of the gravitational potential $V(r) = \int_{\infty}^r Gm/r^2 dr$ and thermostatic potential (specific enthalpy, ejection energy) $H(r) = \int_0^{P(r)} dP/\rho$, (1) and its integrated form

$$-dH/dr = dV/dr, \quad V(r) + H(r) = -\frac{GM}{R}, \quad (2)$$

expresses the conservation of the specific energy as sum of gravitational and internal energies, in a star of mass M and radius R . The two first-order equations (1) can be combined into the second-order Poisson's Law

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dH}{dr} \right) + 4\pi G \rho(H) = 0, \quad (3)$$

in terms of the enthalpy $H(r)$.

The two equations (1) can always be written

$$d \log u / d \log r = 3 - u(r) - n(r)v(r), \quad d \log w / d \log r = u - 1 + v(r) - d \log [1 + n(r)] / d \log r, \quad (4)$$

in terms of the logarithmic derivatives

$$u(r) := d \log m / d \log r, \quad w(r) := n(r)v(r) = -d \log \rho / d \log r, \quad v(r) := -d \log (P/\rho) / d \log r \quad (5)$$

and an index $n(r)$

$$n(r) := d \log \rho / d \log (P/\rho), \quad 1 + 1/n(r) := d \log P / d \log \rho, \quad (6)$$

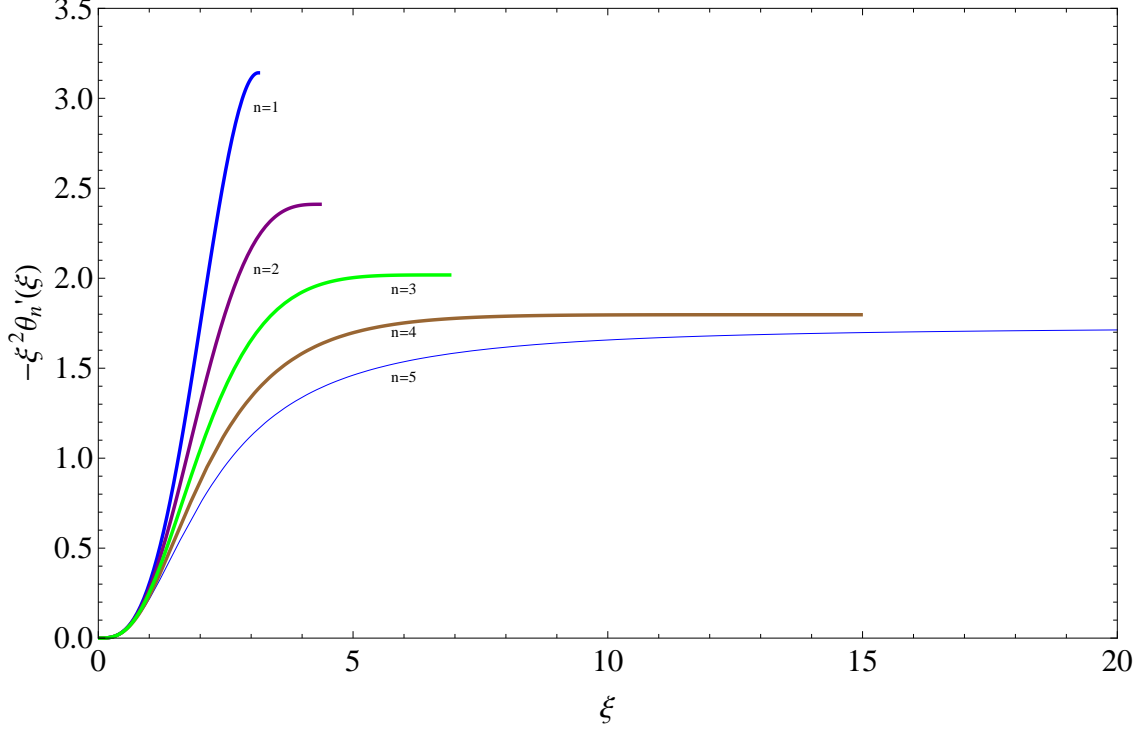


FIG. 2: 'Mass' included inside radius ξ for complete polytropes of order $n=1,2,3,4,5$ terminating at $\xi_{1n} = 3.141, 4.353, 6.897, 14.97, \infty$.

which depends on the local adiabatic equation of state $P = P(\rho)$.

For solutions regular at the origin, spherical symmetry requires that $dP/dr = 0$ and that $\rho(r)$, $P(r)$, $H(r)$ be even functions of r . Mass continuity requires, to order r^2 ,

$$\rho(r) \approx \rho_c(1 - Ar^2), \quad m(r) \approx \frac{4\pi r^3}{3} \cdot (1 - \frac{3}{5}Ar^2) \approx \frac{4\pi r^3}{3} \cdot \rho_c^{2/5} \rho(r)^{3/5} \quad . \quad (7)$$

Thus, near the origin, the average mass density inside radius $\bar{\rho}(r) := \frac{m(r)}{4\pi r^3/3} \approx \rho_c^{2/5} \rho(r)^{3/5}$ and $u(0) = 3, v(0) = 0$. At the stellar surface, $u(R) = 0, v(R) = \infty$. Here $\bar{\rho}(r) := \frac{m(r)}{4\pi r^3/3}$ is the average mass density inside radius, so that $u(r) = 3\rho/\bar{\rho}$ decreases from $u(0) = 3$ at the origin to $u(R) = 0$ at the stellar surface. The ratio $v(r) := \frac{3}{2}(-Gm/r)/(P/\rho) = \frac{3}{2}$ ('gravitational energy')/('internal energy of ideal gas') increases from $v(0) = 0$ at the origin to $v(R) = \infty$ at the stellar surface.

III. SCALE TRANSFORMATIONS ON SOLUTIONS OF THE LANE-EMDEN EQUATION

The symmetry we consider is scaling symmetry, the most general simplification that one can make for any dynamical system. If a scale transformation $r \rightarrow Ar$ transforms m , ρ , P multiplicatively, the logarithmic derivatives u , v will be scale invariant. The structural equations (5) will then be autonomous, if and only if $n = \text{constant}$, so that, $P(r) = K\rho(r)^{1+\frac{1}{n}}$, with the constant K determined by the constant specific entropy. When this is so, scaling symmetry leads to the two coupled autonomous equations for the scale invariants

$$du_n/d\log r = u_n(3 - u_n - nv_n) \quad , \quad dv_n/d\log r = v_n(u_n - 1 + v_n) \quad , \quad (8)$$

implying the first-order equation

$$dv_n/du_n = v_n(u_n - 1 + v_n)/u_n(3 - u_n - nv_n) \quad . \quad (9)$$

This reduced equation for $v_n(u_n)$ incorporates all the consequences of scaling symmetry, from which all the properties of polytropes follow.

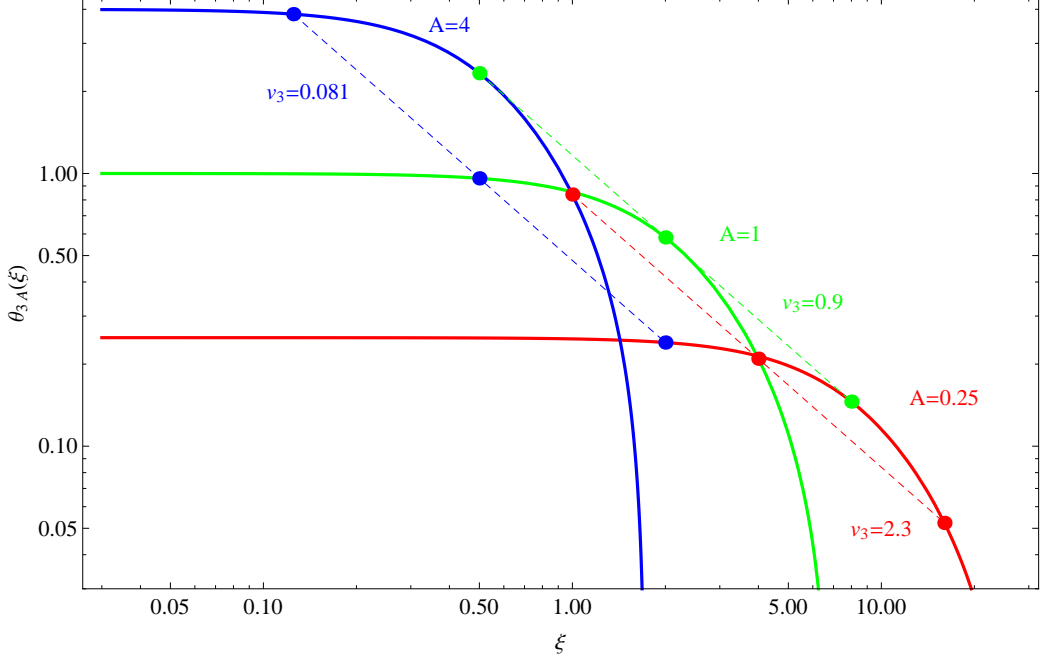


FIG. 3: Emden function $\theta_3(\xi, A)$ for three scales $A=4$ (blue), 1 (green), .25 (red). On a log-log plot, rescaling simply transforms the Lane-Emden functions along lines of constant $v_3 := -d \log(P/\rho)/d \log r$. The three dashed lines connect homologous points for the three illustrative values $v_3 = 0.081, 0.25, 2.3$.

In this and the next section, we consider only *regular solutions* (Emden or E-solutions) of the Lane-Emden equation, which have finite density at the origin, and are applicable only to complete polytropes. At the origin, the initial conditions on regular solutions of 9 are $u_n(0) = 3, v_n(0) = 0$. (We defer to Section IV the irregular solutions $v_n(u_n)$, where the density at the origin is not finite.)

Figure 3 shows how, moving radially outwards, $u_n(r) = 3\rho(r)/\bar{\rho}(r)$ decreases from 3 at the origin to 0 at the outer boundary R and $v_n(r)$ increases from 0 to ∞ . The combinations $\omega_n(r) := (uv_n^n)^{1/(n-1)} \equiv -\xi^{1+\tilde{\omega}_n}\theta'_n$ approach the finite values ${}_0\omega_n(R)$ and characterize each n-polytrope.

In terms of the dimensional central density, pressure ρ_c , $P_c = K\rho_c^{1+1/n}$ and constant $\alpha^2 := ((n+1)/4\pi G)K\rho_c^{1/n-1}$, $H_c := (n+1)(P/\rho)_c \equiv (n+1)K\rho_c^{1/n}$, the dimensional radius, enthalpy, mass density and included mass are

$$r = \alpha\xi, \quad H = H_c\theta_n, \quad \rho = \rho_c\theta_n^n, \quad m(r) = (4\pi\rho_c\alpha^3)(-\xi^2\theta'_n) \quad , \quad (10)$$

where prime designates the derivative $' := d/d\xi$. The scale invariants are

$$u_n := -\xi\theta_n^n/\theta'_n, \quad v_n := -\xi\theta'_n/\theta_n \quad \omega_n := (uv_n^n)^{1/(n-1)} = -\xi^{\frac{n+1}{n-1}} \quad , \quad (11)$$

where $\tilde{\omega}_n := 2/(n-1)$.

In these dimensionless units, , Poisson's Law (3) becomes the *Lane-Emden equation* [6]

$$\frac{d}{d\xi} \left(\xi^2 \frac{d\theta_n}{d\xi} \right) + \xi^2 \theta_n^n = 0, \quad (12)$$

whose regular solutions have

$$\theta_n = \text{const}, \quad \theta'_n(0) = 0, \quad u(0) = 3, \quad v_n(0) = 0 \quad , \quad (13)$$

The *normalized regular solutions* with $\theta_n = 1$ define the *Emden Functions* for the n-polytrope radial evolution (Figure 1). Their dimensionless normalized 'mass' $-\theta_n^2\theta'_n(\xi)$ is shown in Figure 2. The first order equations (8) have analytic solutions listed in Table 1 for $n = 0, 1, 5$, but must be integrated numerically for any other polytropic index n .

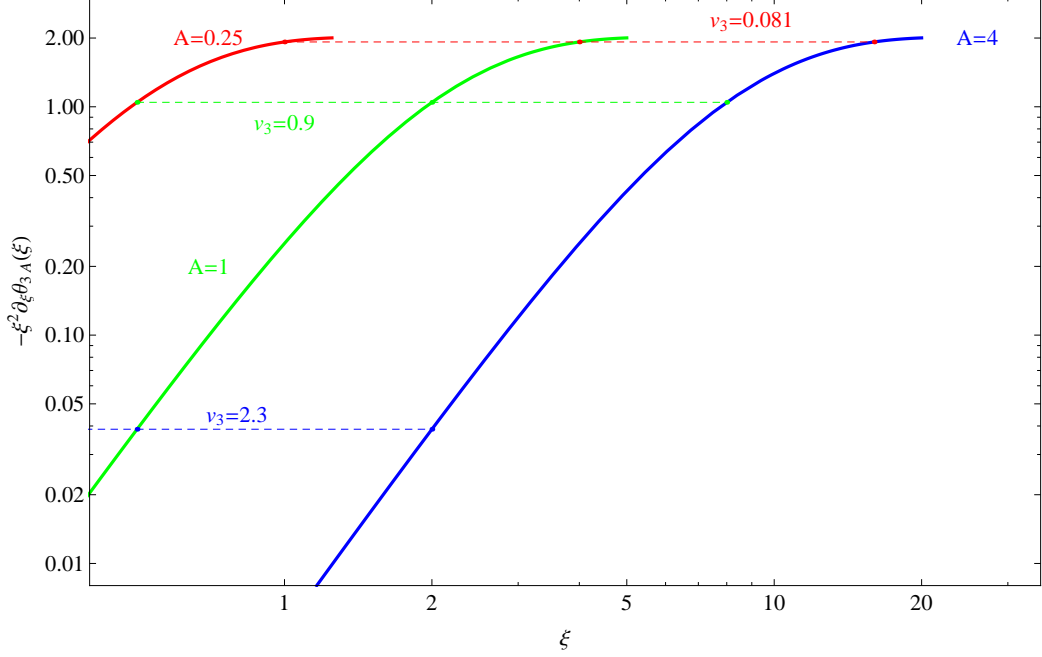


FIG. 4: 'Mass' included inside radius ξ for $n=3$ polytrope. On a log-log plot, rescaling simply transforms the 'mass' along lines of constant $v_3 := -d \log(P/\rho)/d \log r$. The dashed lines connect homologous points for the three illustrative values $v_3 = 0.081, 0.25, 2.3$.

The Lane-Emden equation (12) is invariant under the scale transformation $\xi \rightarrow A\xi$, $t \rightarrow t + \log A$, $\theta_n(\xi) \rightarrow A^{\tilde{\omega}_n} \theta_n(A\xi) := \theta_{nA}(\xi)$. Besides the Emden Function $\theta_n(\xi)$, normalized so that $\theta_{n1}(\xi) \equiv \theta_n(\xi)$, the Lane-Emden equation has rescaled regular solutions $\theta_{nA}(\xi)$, whose value at the origin is A^{ω_n} . Figures 3 and 4 show log-log plots of the rescaled $n=3$ function $\theta_{3A}(\xi)$ and rescaled 'mass' $-\theta'^2 \theta_{3A}(\xi)$ for three different rescalings $A=4$ (blue), 1 (green), 0.25 (red). On a log-log plot, rescaling appears as a translation along the dashed lines of constant v_n and u_n , which connect homologous points. All the familiar properties of polytropes [6] follow from this scaling symmetry incorporated in the reduced equation for $v_n(u_n)$.

TABLE I: Scaling Invariants and normalized Emden Functions for $n=0, 1, 5$

n	$v_n(u_n)$	$u_n(\xi)$	$v_n(\xi)$	$\theta_n(\xi)$	$-\xi^2 \theta'_n(\xi) = \xi^{\frac{(n+1)}{(n-1)}} \omega_n$	ξ_{1n}	${}_0\omega_n$
0	$u = 3$	3	$2\xi^2/(6 - \xi^2)$	$1 - \xi^2/6$	$\xi^3/3$	2.45	0.333
1	parametric $u_n(\xi), v_n(\xi)$	$\xi^2/(1 - \xi \cot \xi)$	$1 - \xi \cot \xi$	$\sin \xi / \xi$	$\sin \xi - \xi \cos \xi$	3.14	\dots
5	$1 - u_5/3$	$3/(1 + \xi^2/3)$	$\xi^2/(3 + \xi^2)$	$(1 + \xi^2/3)^{-1/2}$	$\xi^3/3(1 + \xi^2/3)^{3/2}$	∞	0

IV. PHASE DIAGRAM FOR ALL FIRST INTEGRALS

Equation (9) and the integral curves $v_n(u_n)$ and $u_n(v_n)$ are scale invariant functions of the scale invariant (r/R) plotted in Figure 5 for $n=2, 3, 4, 4.99$. All the dependence on scale R is contained in (8) which reads

$$\log(r/R) = \int_0^\infty \frac{dv_n}{v_n(u_n(v_n) - 1 + v_n)} = \int_3^0 \frac{du_n}{u_n(3 - u_n - nv_n(u_n))} \quad (14)$$

for the *regular solutions* obeying $u_n(0) = 3, v_n(0) = 0$ at the origin.

The *irregular solutions* of (9) apply only to incomplete polytropes, the scale-invariant envelopes of stars. As shown in Figure 6, these are *F-solutions* (green curves, $u_n(0) > 3$) which have infinite density at the origin and *M-solutions* (blue curves, $v_n(0) > 0$) which have density vanishing before the origin ($v(0) > 0$). These irregular solutions would obtain

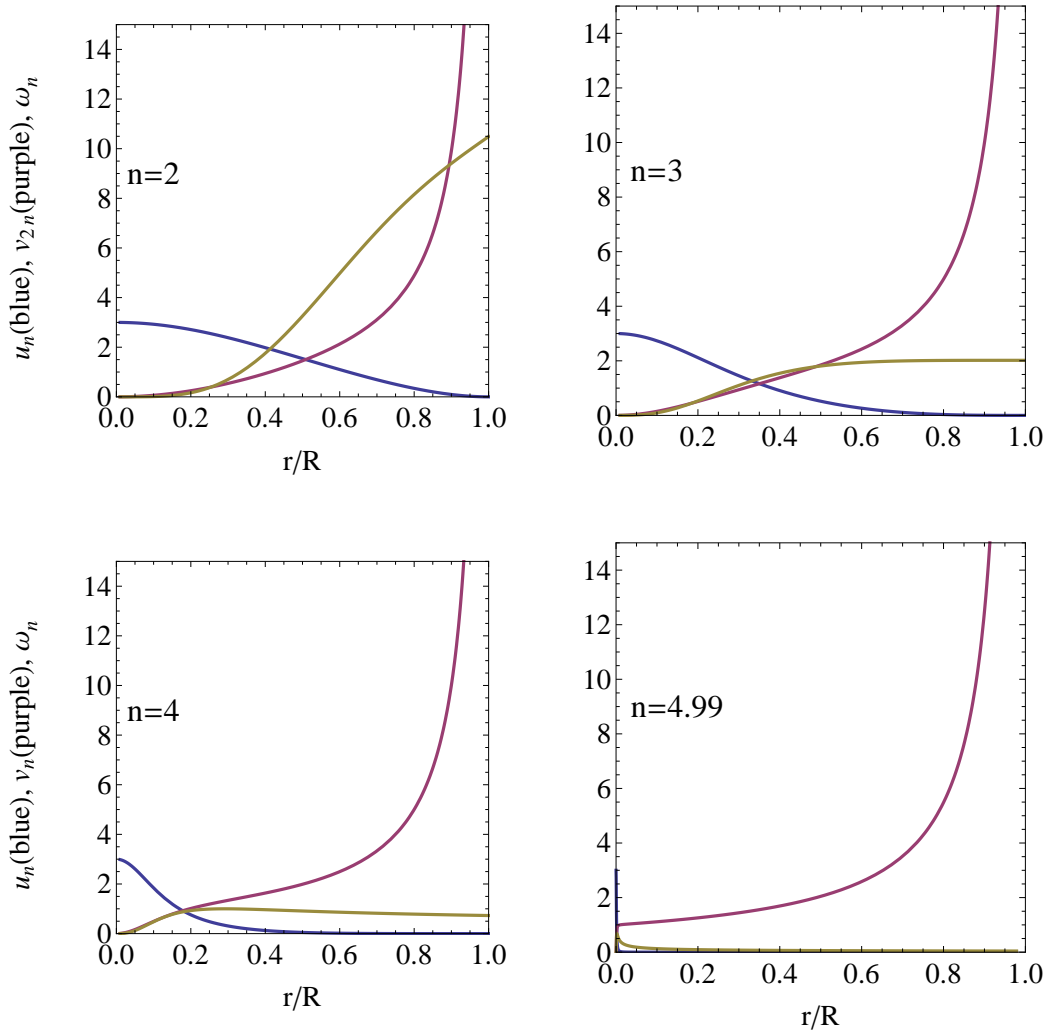


FIG. 5: The Emden scale invariants $u_n(r) = d \log m / d \log r$, $v_n(r) = -d \log P / d \log r$ and $\omega_n(r) = (uv^n)^{1/(n-1)}$. While $v_n(r)$ diverges at the stellar radius R , ω_n (in green) approaches the finite values ${}_0\omega_2 = 10.49$, ${}_0\omega_3 = 2.02$, ${}_0\omega_4 = 0.73$, ${}_0\omega_{4.99} \approx 0$ characteristic of each n -polytrope. The curves in the $n = 4.99$ figure are excellent approximations to the $n = 5$ functions on the bottom line of Table I for which $R = \infty$.

by integrating in from the boundary values ${}_F\omega_n > {}_0\omega_n$ and ${}_0\omega_n > {}_M\omega_n$. Their separatrices (red curves) are the regular (Emden) solutions, which have finite density at the origin, $u_n(0) = 3$, $v_n(0) = 0$ and boundary values ${}_0\omega_n$.

V. CONCLUDING SUMMARY

A symmetry of the equations of motion generally implies, not a conservation law, but a still useful reduction of order of the equations of motion to a first-order equation between invariants, which can then be integrated by quadrature.

For the Lane-Emden equation, the reduced equation is the first-order equation (9), whose first integrals (Figure 6) encapsulate all the properties of index- n polytropes. From this reduced equation (Figure 5), we obtained the regular (Emden) solutions (Figures 1, 2) and simply demonstrated their scale dependence on log-log plots (Figures 3, 4).

-
- [1] P. J. Olver, *Equivalence, Invariants, and Symmetry* (Cambridge University Press, 1995), chapter 7.
 - [2] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems* (John Wiley and Sons, 2001), seventh ed.

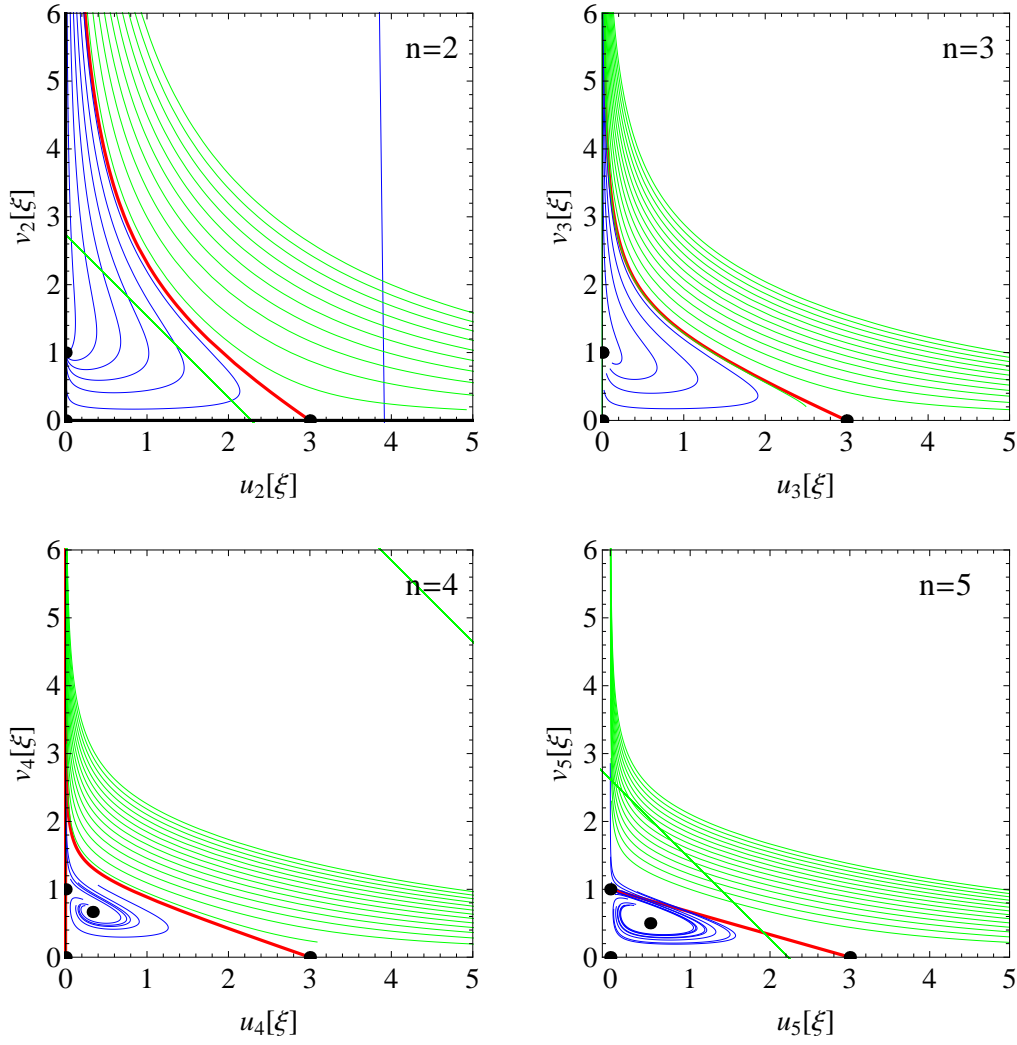


FIG. 6: Critical Points and Phase Diagrams for $v_n(u_n)$ for $n=2, 3, 4, 5$. The green curves (F-solutions) have infinite density at the stellar center. The blue curves (M-solutions) have vanishing density away from the origin. Their separatrix, the red curve (Emden solutions), has finite density at the origin.

- [3] D. W. Jordon and P. Smith, *Nonlinear Ordinary Differential Equations* (Oxford University Press, 1999), 3rd ed., problem 2.13.
- [4] S. Bludman and D. C. Kennedy, J. Math. Phys. **52**, 042092 (2011), arXiv:1008.4086; see also arXiv:1002.4670.
- [5] S. Bludman and D. C. Kennedy, arXiv:1008.4281; see also arXiv:1002.4670.
- [6] S. Chandrasekhar, *An Introduction To The Study Of Stellar Structure* (University of Chicago, 1939), chapters III, IV.